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Tail and Quantile Estimation for
Strongly Mixing Stationary Sequences

by

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Tail and Quantile Estimation for Strongly Mixing Stationary Sequences

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Introduction

The problem of estimating the tail probability $1-F(x) = P(X>x)$ of a r.v. X, for large x has obvious practical importance, for example where large values of X have serious or catastrophic implications for health or safety. In such cases one typically has limited data, so that nonparametric procedures often cannot be successfully applied, but one may also be unwilling to fit a totally parametrized distribution over the entire data range.

A popular compromise with wide applicability is to assume that the tail $1-F(x)$ decays approximately in an exponential manner $e^{-x/\beta}$ as $x \to \infty$ or (by log transformations) as an approximate inverse power law in the sense of regular variation, viz.

$$(1.1) \qquad \frac{1-F(tx)}{1-F(t)} \to x^{-\alpha}, \; x > 0$$

for some index $\alpha > 0$.

Estimation procedures for the exponential or regular variation parameters, based on "high" values in an i.i.d. sample $X_1,\ldots,X_n$ have been studied by a number of authors. In particular the so-called "Hill-estimator" (cf. [10] [7] [2] [6] [11] [1]) is based on the upper $c_n = o(n))$ order statistics $X_j^{(n)}$, $1 \le j \le c_n$, having the form, in the exponential case

$$\hat{\beta}_n = c_n^{-1} \sum_{j=1}^{c_n} (X_j^{(n)} - X_{c_n}^{(n)})$$

and in the regularly varying case is changed by using log X instead of X. One of the main purposes of the present work is to obtain the properties of this and related estimators (and in particular, asymptotic normality) if the observations are no longer independent, but appropriate long range dependence restrictions are assumed, and to extend these results to estimation of tail probabilities and quantiles. We are grateful to T. Hsing for sending us his concurrent work [9] which concerns some of the topics considered here (i.e. the Hill estimate) under similar mixing conditions but using more detailed and precise local dependence assumptions rather than univariate tail conditions as here. Our principal results are given in Sections 4 and 5, following preliminary general central limit results in Sections 2 and 3.

Section 6 proceeds to the original question of estimating tail probabilities $1-F(x)$ for large x, and tail quantiles, i.e. the $(1-p)$th quantile of F for small p values. Asymptotic distributional results are obtained for natural estimates based on the tail parameter estimates of Sections 4 and 5.

In the foregoing results conditions on the dependence structure are obtained so that properties of these estimates for the tail of the marginal d.f. still hold as in the i.i.d. situation. It can also be of interest to estimate tail properties involving not one but groups of the r.v.'s X_i, when it must be expected that the form of the results will also change with introduction of dependence. Such a case is also discussed in Section 6 where tail properties of the maximum $M_N = \max(X_1,\dots,X_N)$ of N consecutive values are considered. In cases when "local dependence" between the X_j is not too high the tail properties of the maximum are the same as in the i.i.d. case. However high local dependence introduces clustering of high values which changes the tail properties of the maximum in a very simple way depending on a single parameter θ known as the "extremal index" of the sequence. In Section 6 the tail parameter estimates are combined with analogously constructed estimates of θ to give estimates of tail probabilities and tail quantiles for the maximum.

The discussion in Sections 3-6 involve exponential tail decay. In Section 7 the modifications needed for regularly varying tails are briefly indicated.

In Section 8 the methods are applied to data consisting of tide heights at a station on the Dutch coast. Estimates are obtained for both tail parameter and extremal index with various choices for the number of upper order statistics used. The effect of these choices provides some insight into the properties of the procedure. In Section 9 simulations are carried out for two processes with exponential tails (i.i.d. and moving average sequences) and two with Pareto tails (moving average and autoregressive sequences). The simulations show that convergence is somewhat slow, and that it might be of interest to investigate "higher order approximations". Nevertheless, the present methods certainly seem sufficient for many engineering problems where ample data is available, such as the water levels from Section 8.

Finally in this introduction we note the precise form of the strong mixing assumption to be used throughout for the (strictly) stationary sequence $X_1, X_2, \ldots$. Write $\mathcal{B}_{ij}$ for the σ-field $\sigma\{X_k: i \leq k \leq j\}$ generated by $X_i, X_{i+1}, \ldots, X_j$ and for fixed n, $\ell < n$,

$$\alpha_{n,\ell} = \sup\{|P(A \cap B) - P(A)P(B)|: A \in \mathcal{B}_{1,k},\ B \in \mathcal{B}_{k+\ell,n},\ 1 \leq k \leq n-\ell\}.$$

Then $\{X_j\}$ will be termed "strongly mixing $(\alpha_{n,\ell}, \ell_n)$" if $\alpha_{n,\ell_n} \longrightarrow 0$ for some $\ell_n = o(n)$. It may be shown that the existence of such a sequence $\{\ell_n\}$ follows if $\alpha_{n,\epsilon n} \longrightarrow 0$ as $n \longrightarrow \infty$ for each $\epsilon > 0$. This "array form" of strong mixing is of course implied by the standard definition (in which $k+\ell$ is not restricted to values no larger than n). For particular purposes weaker forms of the condition may be used - replacing $\mathcal{B}_{ij}$ by the σ-field generated by the functions of $X_i, X_{i+1}, \ldots, X_j$ relevant to the problem (such as $1_{(X_j > u_n)}$ or $(X_j - u_n)_+$ for given u). However in the present context this is unlikely to achieve a significant reduction of conditions and we simply assume the above full strong mixing condition.

2. Notation, assumptions, and general results.

It will be assumed throughout that

(i) $\{X_j\}$ is stationary (with marginal d.f. F), strongly mixing $(\alpha_{n,\ell}, \ell_n)$

(ii) integers $k_n \to \infty$ are chosen such that $k_n(\alpha_{n,\ell_n}+\ell_n/n) \to 0$. Write $r_n = [n/k_n]$. Hence, in particular, $\ell_n=o(r_n)$.

(iii) Integers $c_n \to \infty$ and "levels" u_n are chosen with $(1-F(u_n)) \sim c_n/n$

(iv) ψ is a left-continuous function on the positive real line $R_+ = [0,\infty)$, of bounded variation on finite ranges, and such that $\psi(0) = 0$, $\mathcal{E}\psi^2(X_j) < \infty$.

The conditions (i) - (iv) will be referred to as the *Basic Assumptions*. Other assumptions will be made as needed and stated. For example the condition

(2.1) $c_n = o(k_n)$

will also occasionally be used (when stated). While the main results will be proved without this assumption, its use leads to simplification of sufficient conditions.

Write

$$\beta_n^* = \frac{1}{c_n} \sum_{i=1}^{n} \psi((X_i-u_n)_+)$$

$$\beta_n = \mathcal{E}\beta_n^* = \frac{n}{c_n} \mathcal{E}\psi((X_1-u_n)_+).$$

A primary aim of this section is to show that

(2.2) $(c_n/\lambda_n)^{1/2} (\beta_n^*-\beta_n) \xrightarrow{d} N(0,1)$

where

(2.3) $\lambda_n = \dfrac{k_n}{c_n} \operatorname{var}\{ \sum_{j=1}^{r_n} \psi((X_j-u_n)_+)\}$

under appropriate conditions on F and $\alpha_{n,\ell}$. The dependence of the β_n^* on unknown underlying parameters restricts their practical usefulness as estimators. However it will be seen that the result (2.2) is basic in providing asymptotic distributional results for natural estimators formed by modifying β_n^*.

The proof of (2.2) will be carried out by splitting the sum for β_n^* into groups which may be assumed independent, and applying the Lindeberg Central Limit Theorem. Write $m_n = n-k_nr_n$ and define "intervals" $J_1, J_2 \dots J_{k_n}$ to each consist of r_n consecutive integers, the first m_n+1 being separated by one integer, and the remainder abutting, i.e.,

$$J_i = ((i-1)r_n + i, (i-1)r_n+i+1, \dots, ir_n+i-1), \quad 1 \le i \le m_n$$

$$= ((i-1)r_n+m_n+1, (i-1)r_n+m_n+2, \dots, ir_n+m_n), \quad m_n < i \le k_n$$

Let J_i^* denote the first $(r_n-\ell_n)$ integers in J_i, $1 \le i \le k_n$, and write

$$Y_j \ (= Y_{n,j}) = (X_j-u_n)_+, \qquad 1 \le j \le n$$

$$Z_i \ (= Z_{n,i}) = (\lambda_n c_n)^{-½} \sum_{j\in J_i} \psi(Y_j), \qquad 1 \le i \le k_n$$

$$U_i \ (= U_{n,i}) = (\lambda_n c_n)^{-½} \sum_{j\in J_i^*} \psi(Y_j)$$

$$V_i \ (= V_{n,i}) = Z_i - U_i$$

$$W_i \ (= W_{n,i}) = (\lambda_n c_n)^{-½} \psi(Y_{i(r_n+1)}), \qquad 1 \le i \le m_n.$$

Then $\sum_1^{k_n} Z_i + \sum_1^{m_n} W_i = (\lambda_n c_n)^{-½} \sum_1^n \psi(Y_j)$ so that (2.2) may be rewritten as

$$(2.4) \quad \sum_1^{k_n} (Z_i-\mathcal{E}Z_i) + \sum_1^{m_n} (W_i-\mathcal{E}W_i) \xrightarrow{d} N(0,1).$$

In the following and throughout undesignated ranges for sums and products are to be taken form 1 to k_n.

Lemma 2.1. Assume that the basic assumptions hold and also

(2.5) $k_n(\mathrm{var}V_{n,1} + \mathrm{var}W_{n,1}) \to 0$

Then

(2.6) (i) $\sum_1^{k_n} (V_i - \mathcal{E}V_i) \xrightarrow{P} 0$ (ii) $\sum_1^{m_n} (W_i - \mathcal{E}W_i) \xrightarrow{P} 0.$

Further it then follows that in proving (2.4) the second sum may be omitted and the r.v.'s in the first sum assumed independent. More specifically under the above conditions, (2.4) (and hence (2.2)) hold if and only if

(2.7) $$\Sigma(\hat{Z}_i - \mathcal{E}\hat{Z}_i) \xrightarrow{d} N(0,1)$$

with $\hat{Z}_i$ assumed i.i.d., $\hat{Z}_i \overset{d}{=} Z_1$ $(= (\lambda_n c_n)^{-1/2} \sum_{j=1}^{r_n} \psi(Y_j))$.

Proof: Since the $V_{n,i}$ are defined by groups of X_j which are (for large n) separated by at least ℓ_n, $(r_n/\ell_n \longrightarrow \infty$ by (ii) of the basic assumptions), it follows by a standard induction on the mixing condition (cf. [12]) that

$$|\mathcal{E}\{\exp(it\Sigma(V_j - \mathcal{E}V_j))\} - \Pi\mathcal{E}\{\exp(it(V_j - \mathcal{E}V_j))\}| \le 16k_n\alpha_{n,\ell_n}$$

which tends to zero as $n \to \infty$. Hence in showing (2.6) (i) it may be assumed that the terms are independent. But with this assumption the variance of the sum is k_n var $V_{n,1} \to 0$ by (2.5) so that (2.6) (i) holds. The proof of (2.6) (ii) is entirely similar.

It follows at once that (2.4) holds if and only if

(2.8) $\Sigma(Z_i - \mathcal{E}Z_i) \xrightarrow{d} N(0,1)$.

Now, to prove that (2.7) implies (2.8), let $(\hat{U}_i, \hat{V}_i)$ be pairs having the same distribution as (U_i, V_i) but being independent for $1 \le i \le k_n$. If (2.7) holds, it holds with the specific choice $\hat{Z}_i = \hat{U}_i + \hat{V}_i$ and since clearly (2.6), (i) holds with V_i replaced by $\hat{V}_i$ it follows that $\Sigma(\hat{U}_i - \mathcal{E}\hat{U}_i) \xrightarrow{d} N(0,1)$. But again

$$|\mathcal{E}\{\exp(it\Sigma(U_j - \mathcal{E}U_j)\} - \Pi\mathcal{E}\{\exp(it(U_j - \mathcal{E}U_j))\}| \le 16\, k_n\, \alpha_{n,\ell_n} \to 0$$

and in the second (product) term U_j may be replaced by $\hat{U}_j$ ($\stackrel{d}{=} U_j$) so that $\Sigma(U_i - \mathcal{E}U_i) \xrightarrow{d} N(0,1)$. Finally since $\Sigma Z_i = \Sigma U_i + \Sigma V_i$ it follows from (2.6) that (2.8) and thus (2.4) and finally (2.2) hold. Thus (2.7) implies (2.2). The converse is similarly shown by simply reversing the chain of arguments.

■

This lemma leads at once to a preliminary but useful form of the main result.

<u>Theorem 2.2.</u> Suppose that (2.5) holds, in addition to the basic conditions. Then (2.2) holds if and only if the Lindeberg condition

(2.9) $k_n\ \mathcal{E}\{(Z_{n,1} - \mathcal{E}Z_{n,1})^2\ 1_{(|Z_{n,1} - \mathcal{E}Z_{n,1}| > \epsilon)}\} \to 0$ as $n \to \infty$, each $\epsilon > 0$,

is satisfied.

<u>Proof:</u> This is immediate from Lemma 2.1 and the Lindeberg Central Limit Theorem since $k_n \operatorname{var} Z_{n,1} = 1$. ■

Finally in this section we show that (2.1) provides a simple sufficient condition for (2.5). Less restrictive sufficient conditions will be given later when exponential decay is assumed.

<u>Lemma 2.3.</u> If $\psi(x) \geq 0$ all x and (2.1) holds (i.e. $c_n = o(k_n)$) in addition to the basic conditions, then

$$(2.10) \quad \text{var } Z_{n,1} \sim \mathcal{E}Z_{n1}^2 \quad \text{so that } \lambda_n \sim \lambda_n^* = \frac{k_n}{c_n} \mathcal{E}\{\sum_{j=1}^{r_n} \psi(Y_j)\}^2,$$

and (2.5) holds.

<u>Proof:</u> $$\left[\mathcal{E}\{\sum_1^{r_n} \psi(Y_j)\}\right]^2 = r_n^2 \, (\mathcal{E}\psi(Y_1))^2 = r_n^2(\mathcal{E}(\psi(Y_1)1_{(Y_1>0)}))^2$$

$$\leq r_n^2 \, \mathcal{E}\psi^2(Y_1) \; P\{Y_1>0\} = r_n^2(1-F(u_n)) \; \mathcal{E}\psi^2(Y_1)$$

$$\leq K \, r_n \frac{c_n}{n} \, \mathcal{E}(\sum_1^{r_n} \psi(Y_i))^2$$

since $1-F(u_n) \sim c_n/n$ and $\psi(Y_i) \geq 0$, each i. Since $k_n r_n \sim n$, it thus follows that

$$(\mathcal{E}\sum_1^{r_n} \psi(Y_j))^2 = o\{\mathcal{E}(\sum_1^{r_n} \psi(Y_j))^2\}$$

by (2.1) which yields (2.10).

Since clearly $Z_{n,1} \geq \sum_{i=1}^{[r_n/\ell_n]} V'_{n,i}$ where $V'_{n,i} \overset{d}{=} V_{n,1}$ it follows that $\mathcal{E}Z_{n,1}^2 \geq [r_n/\ell_n] \, \mathcal{E}V_{n,1}^2$ and hence

$$(2.11) \qquad k_n \text{ var } V_{n,1} \leq \frac{k_n}{r_n} \; r_n \, \mathcal{E}V_{n,1}^2 \;\; \leq \; K \frac{k_n}{r_n} \ell_n \, \mathcal{E}Z_{n,1}^2 \sim K \frac{\ell_n}{r_n}$$

by (2.10) since $\text{var } Z_{n,1} = 1/k_n$. But $\ell_n/r_n \sim k_n \, \ell_n/n \to 0$ so that $k_n \text{ var } V_{n,1} \to 0$. Similarly $k_n \text{ var } W_{n,1} \to 0$, showing (2.5). ■

3. Exponentially decreasing tails.

To obtain more detailed results we assume the following exponential-like rate of decay for the tail 1-F(x) of F:

(3.1) $(1-F(t+x))/(1-F(t)) \to e^{-x/\beta}$ as $t \longrightarrow \infty$, all $x \geq 0$, some $\beta > 0$.

Except for the final theorem of this section it will be assumed that the function $\psi(x)$ is non-negative and nondecreasing. We shall refer to the Augmented Basic Assumptions to indicate the addition of these conditions.

Lemma 3.1. If the Augmented Basic Assumptions and (3.1) hold, and if

(3.2) $\int_0^\infty e^{(\epsilon-\beta^{-1})x}\, d\psi(x) < \infty$, some $\epsilon > 0$,

then

(3.3) $\mathcal{E}\psi(Y_{n,1}) \sim \frac{c_n}{n} \int_0^\infty e^{-x/\beta}\, d\psi(x)$.

Proof: $\mathcal{E}\psi(Y_1) = \int_{u_n}^\infty \psi(x-u_n)\, dF(x)$

$$= \int_0^\infty \psi(x)\, dF(u_n + x)$$

$$= \int_0^\infty (1-F(u_n + x)\, d\psi(x)$$

$$\sim (1-F(u_n)) \int_0^\infty e^{-x/\beta}\, d\psi(x)$$

by Theorem 1.8 (ii) of [5]. The result then follows from (iii) of the Basic Assumptions. ■

Note that this result of course holds if it is assumed just that $\mathcal{E}|\psi(Y_1)| < \infty$ rather than $\mathcal{E}\psi^2(Y_1) < \infty$ in the Basic Assumptions. Using this, the lemma yields the following simple but useful facts.

Lemma 3.2. Under the assumptions of Lemma 3.1, with λ_n^* as in (2.10),

(3.4) $k_n \mathcal{E}Z_{n,1} \sim A(c_n/\lambda_n)^{1/2}$, $A = \int_0^\infty e^{-x/\beta}\, d\psi(x)$

(3.5) $k_n \mathcal{E}Z_{n,1}^2 = 1 + A^2\frac{c_n}{\lambda_n k_n}(1+o(1)) \leq K(1 + \frac{c_n}{\lambda_n k_n})$.

If also (3.2) holds with ψ^2 replacing ψ, then

$$(3.6)\quad \liminf \lambda_n^* = \liminf \frac{k_n}{c_n} \mathcal{E}(\sum_1^{r_n} \psi(Y_j))^2 \geq \int_0^\infty e^{-x/\beta}\, d\psi^2(x)$$

and

$$(3.7)\quad k_n \mathcal{E}W_{n,1}^2 \leq k_n \mathcal{E}V_{n,1}^2 \leq K \min(\frac{c_n \ell_n}{n \lambda_n}, \frac{\ell_n^2}{r_n \lambda_n}) + o(1)$$

Proof: (3.4) follows at once from (3.3) since $r_n \sim n/k_n$, and (3.5) is obtained from (3.4) by noting that var $Z_n = 1/k_n$.

If $\psi(x) \geq 0$ then $\mathcal{E}(\sum_1^{r_n} \psi(Y_j))^2 \geq r_n \mathcal{E}\psi^2(Y_1)$ so that $\liminf \lambda_n^* \geq \liminf k_n c_n^{-1} r_n \mathcal{E}\psi^2(Y_1)$, giving (3.6) by (3.3) with ψ^2 for ψ. Finally as in the proof of Lemma 2.3, by (2.11), and using (3.5),

$$k_n \mathcal{E}W_{n1}^2 \leq k_n \mathcal{E}V_{n,1}^2 \leq K \frac{k_n \ell_n}{r_n} \mathcal{E}Z_{n,1}^2 \leq K \frac{\ell_n}{r_n}(1 + \frac{c_n}{\lambda_n k_n}) \leq K \frac{c_n \ell_n}{n \lambda_n} + o(1)$$

since $\ell_n/r_n \sim k_n \ell_n/n \to 0$. The second bound $k_n \mathcal{E}V_{n1}^2 \leq K \ell_n^2/(r_n \lambda_n)$ follows at once from the obvious (Minkowski) inequality $\mathcal{E}V_{n,1}^2 \leq \ell_n^2 \mathcal{E}\psi^2(Y_{n1})/(c_n \lambda_n)$ and (3.3) with ψ^2 in place of ψ. ∎

The conditions (2.5) used in Theorem 2.2 may be readily verified directly in particular cases. However simple sufficient conditions are obtainable from (3.7), viz either

$$(3.8)\quad c_n \ell_n / (n \lambda_n) \to 0$$

or

$$(3.9)\quad \ell_n^2/(r_n \lambda_n) \to 0$$

The (more useful) condition (3.8) is implied in particular by (2.1), viz $c_n = o(k_n)$, by Lemma 2.3, (3.6) and the basic assumption $k_n \ell_n/n \to 0$.

To give simple sufficient conditions for the Lindeberg criterion it is convenient to truncate $\psi(Y_j)$ as follows. For constants w_n to be specified, define

$$Y_j' = Y_j \, 1_{(Y_j \le w_n)} + w_n \, 1_{(Y_j > w_n)}, \quad 1 \le j \le n$$

$$Z_1' = (c_n\lambda_n)^{-1/2} \sum_{j=1}^{r_n} \psi(Y_j')$$

<u>Lemma 3.3.</u> Let the Augmented Basic Assumptions and (3.1) hold, and for some $0 < \epsilon < 1$, let w_n satisfy

$$(3.10) \quad \lambda_n^{-1} r_n \int_{w_n}^{\infty} \exp\{(\epsilon-\beta^{-1})x\} d\psi^2(x) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Then $k_n \mathcal{E}(Z_1-Z_1')^2 \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof: Note, using Minkowski's Inequality, that

$$\begin{aligned} r_n^{-2} c_n \lambda_n \, \mathcal{E}(Z_1-Z_1')^2 &= r_n^{-2} \mathcal{E}\{\sum_{j=1}^{r_n} (\psi(Y_j)-\psi(w_n)) 1_{(Y_j>w_n)}\}^2 \\ &\le \mathcal{E}\{(\psi(Y_1) - \psi(w_n))^2 1_{(Y_1>w_n)}\} \\ &\le \mathcal{E}\{\psi(Y_1)^2 - \psi(w_n)^2) 1_{(Y_1>w_n)}\} \\ &= \int_{u_n+w_n}^{\infty} (\psi(x-u_n)^2-\psi(w_n)^2) \, dF(x) \\ &= \int_{w_n}^{\infty} (1-F(y+u_n)) \, d\psi^2(y) \\ &\le (1+\epsilon)(1-F(u_n)) \int_{w_n}^{\infty} \exp[(\epsilon-\beta^{-1})x] d\psi^2(x) \end{aligned}$$

by Proposition 1.7 of [5], from which the desired conclusion follows by (3.10) since $1-F(u_n) \sim c_n/n$ and $r_n k_n \sim n$. ∎

The following theorems are now simply obtained.

<u>Theorem 3.4</u> Let the Augmented Basic Assumptions, (2.5), (3.1), (3.2) all hold, and let w_n satisfy (3.10) and

$$(3.11) \quad (c_n\lambda_n)^{-1/2} \, r_n \, \psi(w_n) \to 0.$$

Then (2.2) holds, i.e. $(c_n/\lambda_n)^{1/2}\ (\beta_n^*-\beta_n) \xrightarrow{d} N(0,1)$.

Proof: By Theorem 2.2 it is sufficient to show that the Lindeberg Condition (2.9) holds. Now if X,Y are any two random variables, it is readily checked that

(3.12) $$(X+Y)^2\ 1_{(|X+Y|\geq\epsilon)} \leq 4(X^2 1_{(|X|\geq\epsilon/2)} + Y^2 1_{(|Y|\geq\epsilon(2))})$$

from which it follows (with $Z_1'-\mathcal{E}Z'$ for X and $(Z_1-\mathcal{E}Z_1) - (Z_1'-\mathcal{E}Z_1')$ for Y) that

$$k_n\mathcal{E}\{(Z_1-\mathcal{E}Z_1)^2 1_{(|Z_1-\mathcal{E}Z_1|>\epsilon)}\} \leq 4k_n\mathcal{E}\{(Z_1'-\mathcal{E}Z_1')^2 1_{(|Z_1'-\mathcal{E}Z_1'|>\epsilon/2)}\} + 4k_n\mathcal{E}(Z_1-Z_1')^2.$$

The first term on the right tends to zero trivially since $0\leq Z_1'\leq(c_n\lambda_n)^{-1/2}r_n\psi(w_n) \longrightarrow 0$ by (3.11), and the last term tends to zero by Lemma 3.3, so that (2.9) holds, as desired. ∎

In the final result of this section we generalize Theorem 3.4 to include functions $\psi(x)$ which can be negative and not necessarily monotone.

Theorem 3.5 Suppose the assumptions of Theorem 3.4 are satisfied for each of the functions $\psi_1(x)$, $\psi_2(x)$ and write $\psi(x) = \alpha_1\psi_1(x) + \alpha_2\psi_2(x)$, α_1, α_2 (positive or negative) constants. Let $\lambda_n^{(1)}$, $\lambda_n^{(2)}$, λ_n be defined as in (2.3) relative to ψ_1, ψ_2, ψ respectively, and suppose that $\lambda_n^{(k)} \leq K\lambda_n$ k=1,2, n=1,2,3,... . Then (2.2) holds, i.e. $\beta_n^* = c_n^{-1}\sum_1^n \psi(X_i-u_n)_+$, $\beta_n = n\ c_n^{-1}\ \mathcal{E}\psi(X_1-u_n)_+)$, satisfy

$$(c_n/\lambda_n)^{1/2}\ (\beta_n^*-\beta_n) \xrightarrow{d} N(0,1).$$

Proof: If $\beta_n^{*(k)} = c_n^{-1}\sum_{i=1}^n \psi_k((X_i-u_n)_+)$, $\beta_n^{(k)} = \mathcal{E}\beta_n^{*(k)}$, k=1,2 then Theorem 3.4 shows that $(c_n/\lambda_n^{(k)})^{1/2}\ (\beta_n^{*(k)}-\beta_n^{(k)}) \xrightarrow{d} N(0,1)$ and hence by Theorem 2.2 the Lindeberg conditions

(3.14) $k_n \mathscr{E}\{(Z_{n,1}^{(k)} - \mathscr{E}Z_{n,1}^{(k)})^2 1_{\{|Z_{n,1}^{(k)} - \mathscr{E}Z_{n,1}^{(k)}| \geq \epsilon\}}\} \to 0$ as $n \to \infty$, each $\epsilon > 0$

hold, where $Z_{n,1}^{(k)} = (\lambda_n^{(k)} c_n)^{-\frac{1}{2}} \sum_{j \in J_1} \psi_k(Y_j)$, $k = 1,2$.

Since $\lambda_n^{(k)} \leq K\lambda_n$ this Lindeberg Condition continues to hold for each $k = 1,2$ if $\lambda_n^{(k)}$ is replaced by λ_n in the definition of $Z_{n,1}^{(k)}$ and hence it holds for $\alpha_1 Z_{n,1}^{(1)} + \alpha_2 Z_{n,1}^{(2)}$ by the inequality (3.12). The remaining conditions of Theorem 2.2 regarding ψ are readily checked, giving the stated result. ■

4. The Hill Estimator

Let

$$N_n(x) = \sum_{i=1}^{n} 1_{\{X_i \geq x\}}$$

be the number of exceedances of x by $X_1, \ldots, X_n$, and let $\{z_n\}$ be a sequence of "levels", non-random or random. The Hill estimator $\hat{\beta}_n$ is then defined by

$$\hat{\beta}_n = \hat{\beta}_n(z_n) = \frac{1}{N_n(z_n)} \sum_{i=1}^{n} (X_i - z_n)_+ .$$

The two cases which mainly have been considered are $z_n = u_n$, with $\{u_n\}$ a given non-random sequence, and $z_n = X_{c_n}^{(n)}$, the c_n-th largest of $X_1, \ldots, X_n$, with $\{c_n\}$ a given non-random sequence of integers. This leads to the two estimators

$$\hat{\beta}_n(u_n) = \frac{1}{N_n(u_n)} \sum_{i=1}^{n} (X_i - u_n)_+ \tag{4.1}$$

and

$$\begin{aligned} \hat{\beta}_n(X_{c_n}^{(n)}) &= \frac{1}{c_n} \sum_{i=1}^{c_n} (X_i - X_{c_n}^{(n)})_+ \\ &= \frac{1}{c_n} \sum_{i=1}^{c_n} (X_i^{(n)} - X_{c_n}^{(n)}) . \end{aligned} \tag{4.2}$$

For the present purposes a somewhat stronger tail condition than before is needed. We suppose F has one derivative F' which satisfies

(4.3) $$\lim_{t\to\infty} \frac{F'(t)}{1-F(t)} = \frac{1}{\beta} .$$

Further, for $\{u_n\}$ (or $\{c_n\}$) given, define $\{c_n\}$ (or $\{u_n\}$) by

(4.4) $$\frac{n}{c_n} (1-F(u_n)) = 1,$$

and assume throughout that $c_n \to \infty$, $c_n/n \to 0$. Here, if u_n is given, the c_n obtained from (4.4) may not be an integer. However, in that case we replace c_n by its integer part. It is straightforward to check that this does not affect the proofs below.

We will prove that the estimators in (4.1) and (4.2) are asymptotically normal, with means

(4.5) $$\beta_n = \frac{n}{c_n} \mathcal{E}(X_1 - u_n)_+$$

$$= \mathcal{E}\{(X_1 - u_n) | X_1 > u_n\}.$$

It follows from (3.1) (which in turn is implied by (4.3), cf. Lemma A2 of the appendix), as in Lemma 3.1, that

(4.6) $$\beta_n \to \beta < \infty .$$

Let

$$\bar{F}(x) = 1-F(x) = \mathcal{E}1_{\{X_1>x\}}$$

and define, with notation similar to that in Section 2,

(4.7) $$\lambda_n = \frac{k_n}{c_n} \operatorname{Var} \sum_{j=1}^{r_n} \{(X_j - u_n)_+ - \beta\, 1_{\{X_j>u_n\}}\} .$$

Further, let

$$S_n(x) = \frac{N_n(x) - n\,\bar{F}(x)}{\sqrt{\lambda_n\, c_n}}, \qquad E_n(x) = \hat{\beta}_n(x) - \beta_n .$$

<u>Lemma 4.1</u> (i) Suppose (4.3) holds, $c_n/\lambda_n \to \infty$,

(4.8) $$\sqrt{\frac{c_n}{\lambda_n}}\left[\frac{1}{c_n}\sum_{i=1}^{n}(X_i - u_n)_+ - \beta_n\right] - \beta\, S_n(u_n) \xrightarrow{d} N(0,1)$$

and either the first or the second term in (4.8) is tight (so that both are tight). Then

(4.9) $$\sqrt{\frac{N_n(u_n)}{\lambda_n}}\, E_n(u_n) - \left\{\sqrt{\frac{c_n}{\lambda_n}}\left[\frac{1}{c_n}\sum_{i=1}^{n}(X_i - u_n)_+ - \beta_n\right] - \beta\, S_n(u_n)\right\} \xrightarrow{P} 0$$

and

(4.10) $$\sqrt{\frac{N_n(u_n)}{\lambda_n}}\, E_n(u_n) \xrightarrow{d} N(0,1).$$

(ii) If furthermore

(4.11) $$\sqrt{\frac{c_n}{\lambda_n}}\,(z_n - u_n) \quad \text{is tight,}$$

and

(4.12) $$S_n(z_n) - S_n(u_n) \xrightarrow{P} 0,$$

then

$$\sqrt{\frac{N_n(z_n)}{\lambda_n}}\, E_n(z_n) - \sqrt{\frac{N_n(u_n)}{\lambda_n}}\, E_n(u_n) \xrightarrow{P} 0,$$

so that also

$$\sqrt{\frac{N_n(z_n)}{\lambda_n}}\, E_n(z_n) \xrightarrow{d} N(0,1).$$

<u>Proof</u>. Since $c_n = n\,\bar{F}(u_n)$, by (4.4), we have that

$$(4.13)\quad \sqrt{\frac{c_n}{\lambda_n}}\, E_n(u_n) = \sqrt{\frac{c_n}{\lambda_n}} \left[\frac{1}{c_n} \sum_{i=1}^{n} (X_i - u_n)_+ - \beta_n \right] - S_n(u_n)\, \hat{\beta}_n(u_n) .$$

Further, since $c_n/\lambda_n \to \infty$, $\beta_n \to \beta$, and the two terms in (4.8) are tight, it follows that

$$(4.14)\quad \frac{1}{c_n} N_n(u_n) \xrightarrow{P} 1, \qquad \frac{1}{c_n} \sum_{i=1}^{n} (X_i - u_n)_+ \xrightarrow{P} \beta$$

and hence also $\hat{\beta}_n(u_n) \xrightarrow{P} \beta$. Now (4.9) and hence (4.10) readily follow using (4.8), (4.14) and tightness of $S_n(u_n)$.

It is obvious that if (3.1) holds, then $\bar{F}(z_n)/\bar{F}(u_n) \to 1$ if $z_n - u_n \to 0$. However, since (4.3) implies (3.1) by Lemma A2 of the appendix, this follows from (4.3). Similar arguments to those above now show that if in addition (4.12) holds, then

$$(4.15)\qquad \frac{1}{c_n} N_n(z_n) \xrightarrow{P} 1.$$

Thus, to establish the rest of the lemma, it is enough to show that

$$\sqrt{\frac{c_n}{\lambda_n}}\, \{\hat{\beta}_n(z_n) - \hat{\beta}_n(u_n)\} \xrightarrow{P} 0.$$

We will first bound this expression on the set $\{z_n > u_n\}$. Formally, this may be done by multiplying by $1_{\{z_n > u_n\}}$ throughout, but for simplicity of notation we will just assume $z_n > u_n$ in the computations below. Then,

$$\sum_{i=1}^{n} (X_i - u_n)_+ = \sum_{i=1}^{n} (X_i - z_n)_+ + (z_n - u_n)\, N_n(z_n) + r_n,$$

for $r_n = \sum_{i=1}^{n} (X_i - u_n)\, 1_{\{u_n \le X_i < z_n\}}$.

By straightforward computations,

$$\sqrt{\frac{c_n}{\lambda_n}}\,\{\hat{\beta}_n(z_n)-\hat{\beta}_n(u_n)\} = \frac{c_n}{N_n(z_n)}\,(S_n(u_n) - S_n(z_n))\,\hat{\beta}_n(u_n)$$

$$+ \sqrt{\frac{c_n}{\lambda_n}}\left[\frac{n(\bar{F}(u_n)-\bar{F}(z_n))}{N_n(z_n)}\,\hat{\beta}_n(u_n) + u_n-z_n\right] + \sqrt{\frac{c_n}{\lambda_n}}\,\frac{r_n}{N_n(z_n)}$$

$$= A_n + B_n + C_n, \quad \text{say.}$$

It follows directly from (4.15), (4.12) and $\hat{\beta}_n(u_n) \overset{P}{\to} \beta$ that $A_n \overset{P}{\to} 0$ as $n \to \infty$.

Further, since $\bar{F}(x) = 1-F(x)$, it follows from (4.3) that

$$\log \frac{\bar{F}(z_n)}{\bar{F}(u_n)} = \int_0^{z_n-u_n} \frac{-F'(u_n+s)}{1-F(u_n+s)}\,ds$$

$$= -(z_n-u_n)/\beta + o(z_n-u_n)\ .$$

According to Taylor's formula, $1-x = -\log x + o(\log x)$, as $\log x \to 0$, and hence, using (4.4) and $\bar{F}(z_n)/\bar{F}(u_n) \to 1$,

(4.16) $$n(\bar{F}(u_n) - \bar{F}(z_n)) = c_n\left[1 - \frac{\bar{F}(z_n)}{\bar{F}(u_n)}\right]$$

$$= c_n(z_n-u_n)/\beta + o(c_n(z_n-u_n))\ .$$

Thus,

$$B_n = \sqrt{c_n/\lambda_n}\ \{N_n(z_n)^{-1}\,[c_n(z_n-u_n)/\beta + o(c_n(z_n-u_n)]\hat{\beta}_n(u_n) + u_n-z_n\}$$

$$= \sqrt{\frac{c_n}{\lambda_n}}\ (z_n-u_n)\left\{\frac{c_n\,\hat{\beta}_n(u_n)}{\beta\,N_n(z_n)}\ (1 + o(1)) - 1\right\} \xrightarrow{P} 0\ ,$$

by (4.11) and (4.15), since $\hat{\beta}_n(u_n) \overset{P}{\to} \beta$.

Finally, to show that $C_n \overset{P}{\to} 0$ it suffices by (4.15) to prove that

(4.17) $$\frac{r_n}{\sqrt{\lambda_n\,c_n}} \overset{P}{\to} 0\ .$$

From (4.11), it follows that we may choose z_n' non-random, with $P(z_n' > z_n) \to 1$ and $(c_n/\lambda_n)^{1/4}(z_n' - u_n) \to 0$. Since r_n is increasing in z_n, it is hence enough to prove (4.17) with z_n replaced by z_n'. It then follows from (4.16) that

$$\varepsilon \frac{r_n}{\sqrt{\lambda_n c_n}} \le \frac{(z_n'-u_n)\, n(\bar{F}(u_n) - \bar{F}(z_n'))}{\sqrt{\lambda_n c_n}}$$

$$= O\left[\sqrt{\frac{c_n}{\lambda_n}}\,(z_n' - u_n)^2\right] \to 0 \quad \text{as } n \to \infty .$$

Hence also $C_n \xrightarrow{P} 0$, and the desired conclusion holds on the set $\{z_n > u_n\}$. Similar considerations on $\{z_n < u_n\}$ then conclude the proof. ∎

It seems likely that under conditions similar to those in Section 2, the sequence of processes $\{S_n(u_n + x);\ |x| \le 1\}$ is tight, and has a continuous limit in $D[-1,1]$. In that case (4.12) would hold for any sequence $\{z_n\}$ with $z_n - u_n \xrightarrow{P} 0$. However, here we will consider only $z_n = X_{c_n}^{(n)}$.

<u>Lemma 4.2</u> Suppose $S_n(u_n) \xrightarrow{d} Z$, for some random variable Z, and that (4.12) holds for any non-random sequence $\{z_n\}$ with $\sqrt{c_n/\lambda_n}\,(z_n - u_n)$ bounded. Then (4.12) holds also for $z_n = X_{c_n}^{(n)}$, i.e.

(4.18) $$S_n(X_{c_n}^{(n)}) - S_n(u_n) \xrightarrow{P} 0$$

and

(4.19) $$\sqrt{\frac{c_n}{\lambda_n}}\,(X_{c_n}^{(n)} - u_n) - \beta S_n(u_n) \xrightarrow{P} 0.$$

<u>Proof</u>. By definition $N_n(X_{c_n}^{(n)}) = c_n$ and hence

$$S_n(X^{(n)}_{c_n}) = \frac{c_n - n\bar{F}(X^{(n)}_{c_n})}{\sqrt{c_n \lambda_n}} .$$

Let $U = (1/(1-F))^{\leftarrow}$ be the right continuous inverse of $1/(1-F) = 1/\bar{F}$ and set, for $x \in \mathbb{R}$,

$$z_n = U\left(\frac{n}{c_n} \frac{1}{1 - x\sqrt{\lambda_n/c_n}} \right)$$

so that

$$\text{(4.20)} \qquad \{ S_n(X^{(n)}_{c_n}) \le x \} = \{ X^{(n)}_{c_n} \le z_n \} .$$

Also $u_n = U(n/c_n)$ and hence

$$(z_n - u_n) = \int_1^{1/(1-x\sqrt{\lambda_n/c_n})} \frac{n}{c_n} s\, U'\left[\frac{n}{c_n} s \right] \frac{ds}{s} .$$

By (i) of the remark after Lemma A2 of the appendix, (4.3) implies that

$$\frac{n}{c_n} s\, U'\left[\frac{n}{c_n} s \right] \to \beta , \quad n \to \infty,$$

uniformly for s in the considered range, and thus

$$\sqrt{\frac{c_n}{\lambda_n}} (z_n - u_n) \sim \sqrt{\frac{c_n}{\lambda_n}} \beta\left[-\log \left(1 - x\sqrt{\frac{\lambda_n}{c_n}}\right) \right] ,$$

which clearly is bounded, so that (4.12) holds for this $X^{(n)}_{c_n}$.

By (4.20) and the definition of $X^{(n)}_{c_n}$

$$\left\{ S_n(X^{(n)}_{c_n}) \le x \right\} = \{N_n(z_n) < c_n\}$$

$$= \left\{ S_n(z_n) \le \frac{c_n - n\,\bar{F}(z_n)}{\sqrt{c_n \lambda_n}} \right\}$$

$$= \{S_n(z_n) \le x\} \ .$$

Since $S_n(u_n) \xrightarrow{d} Z$ it follows from (4.12) that

$$(S_n(X_{c_n}^{(n)}),\ \ S_n(u_n)) \xrightarrow{d} (Z,\ Z) \ ,$$

which implies $S_n(X_{c_n}^{(n)}) - S_n(u_n) \xrightarrow{d} Z\text{-}Z = 0$, proving (4.18).

To prove (4.19), set

$$z_n = u_n + x\sqrt{\lambda_n / c_n},$$

so that

$$\left\{\sqrt{\frac{c_n}{\lambda_n}}\,(X_{c_n}^{(n)} - u_n) \le x\right\} = \{X_{c_n}^{(n)} \le z_n\} = \{N_n(z_n) < c_n\}$$

$$= \left\{\beta S_n(z_n) \le \frac{\beta}{\sqrt{\lambda_n c_n}}\,(c_n - n\,\bar{F}(z_n)) \right\} .$$

Here $c_n = n\,\bar{F}(u_n)$, by (4.4), and it follows from (4.16) that

$$\frac{\beta}{\sqrt{\lambda_n c_n}}\,(c_n - n\,\bar{F}(z_n)) \sim \frac{\beta}{\sqrt{\lambda_n c_n}}\,c_n(z_n - u_n)/\beta = x \ .$$

Since $\beta S_n(z_n)$ converges in distribution, reasoning as in the first part of the proof shows that (4.19) holds. ■

Asymptotic normality of the Hill estimators (4.1) and (4.2) now follows from the results of Sections 2 and 3.

<u>Theorem 4.3</u> (i) Suppose (4.3) holds, $c_n / \lambda_n \to \infty$, and the conditions of Theorem 3.5 are satisfied for

$$\psi_1(x) = x\, 1_{\{x\geq 0\}}, \quad \psi_2(x) = 1_{\{x\geq 0\}}.$$

Then

$$\sqrt{\frac{N_n(u_n)}{\lambda_n}}\; E_n(u_n) \xrightarrow{d} N(0,1) \ . \tag{4.21}$$

(ii) If the assumptions of part (i) are satisfied and if, writing $I_i = [u_n, z_n)$ if $z_n > u_n$, and $[z_n, u_n)$ otherwise,

$$\frac{1}{\lambda_n}\frac{k_n}{c_n}\,\mathrm{var}\Big[\sum_{i=1}^{r_n} 1_{\{X_i \in I_n\}}\Big] \to 0, \tag{4.22}$$

for any non-random $\{z_n\}$ with $\sqrt{c_n/\lambda_n}\,(z_n-u_n)$ bounded, then

$$\sqrt{\frac{N_n(X^{(n)}_{c_n})}{\lambda_n}}\; E_n(X^{(n)}_{c_n}) - \sqrt{\frac{N_n(u_n)}{\lambda_n}}\; E_n(u_n) \xrightarrow{P} 0 \ ,$$

and hence

$$\sqrt{\frac{N_n(X^{(n)}_{c_n})}{\lambda_n}}\; E_n(X^{(n)}_{c_n}) \xrightarrow{d} N(0,1) \ . \tag{4.23}$$

<u>Proof</u>. (i) Setting $\alpha_1=1$, $\alpha_2=-\beta$ in Theorem 3.5, it follows that (4.8) holds. Since also the other conditions of Lemma 4.1 (i) are satisfied, (4.21) follows.

(ii) Clearly

$$|S_n(z_n) - S_n(u_n)| = \Big|\, \frac{1}{\sqrt{c_n\,\lambda_n}} \Big[\sum_{i=1}^{n} 1_{\{X_i \in I_n\}} - n\, \mathcal{E}\, 1_{\{X_1 \in I_n\}}\Big] \Big|,$$

and hence, by Lemmas 4.2, 4.1 the result follows if we prove that the righthand side of the expression above tends to zero in probability.

Proceeding along similar, but somewhat cruder lines than in Lemma 2.1, split the integers between 1 and n up into $[n/k_n]$ "intervals" of length r_n, with one shorter interval remaining. As in Lemma 2.1 the sums of the $1_{\{X_i \in I_n\}}$ for i belonging to the first, third, ... interval (the "odd intervals") are asymptotically independent, and it hence follows from (4.22) that the sum over all i belonging to the odd intervals tends to zero. Similarly, the sum over all i belonging to "even intervals" tends to zero.

Finally,

$$\mathcal{E}|1_{\{X_i \in I_n\}} - \mathcal{E}1_{\{X_i \in I_n\}}| \le \mathcal{E}1_{\{X_i \in I_n\}} = |\bar{F}(z_n) - \bar{F}(u_n)| \sim c_n n^{-1}|z_n - u_n|/\beta,$$

by (4.16). Thus the expectation of the sum over i belonging to the "remaining short interval" is bounded by

$$r_n \frac{c_n}{n\sqrt{c_n \lambda_n}} |z_n - u_n| \to 0,$$

since $r_n/n \to 0$ and $\sqrt{c_n/\lambda_n}\,(z_n - u_n)$ is bounded. This completes the proof of part (ii). ■

5. Estimation of λ_n

For inference purposes it is of course desirable to estimate the basic unknown variance λ_n. Natural estimators are given by

$$(5.1)\quad \hat{\lambda}_n = (N_n(z_n))^{-1} \sum_{i=1}^{k_n} \{ \sum_{j \in I_i} [(X_j - z_n)_+ - \hat{\beta}_n 1_{(X_j > z_n)}]\}^2$$

where I_i is the interval $((i-1)r_n+1, \ldots ir_n)$ and z_n is either the nonrandom level u_n or the random level $X_n^{(c_n)}$. Here for simplicity we consider the former case and show that in Theorem 4.3 λ_n may be replaced by the estimator $\hat{\lambda}_n$ under appropriate conditions. This will clearly be the case if $\hat{\lambda}_n/\lambda_n \overset{P}{\to} 1$ which will be shown to hold at least for sequences $\{c_n\}$ satisfying further conditions including a strengthening of (3.8), viz.

(5.2) $c_n = o(k_n \lambda_n)$.

Note that since $N_n(u_n)/c_n \xrightarrow{P} 1$ the divisor $N_n(u_n)$ in (4.1) may be replaced by c_n so that it is sufficient to show that

$$(5.3) \quad (c_n\lambda_n)^{-1} \sum_i \{ \sum_{j\in I_i} [(X_j-u_n)_+ - \beta 1_{(X_j>u_n)}] - (\hat{\beta}_n-\beta)N_n(I_i)\}^2 \xrightarrow{P} 1$$

where $N_n(I_i) = \sum_{j\in I_i} 1_{(X_j>u_n)}$. Using the notation of Section 2 with

$\psi(x) = x_+ - \beta 1_{(x>0)}$, (5.3) becomes

$$(5.4) \quad \Sigma[Z_i - (c_n\lambda_n)^{-1/2}(\hat{\beta}_n-\beta)\ N_n(I_i)]^2 \xrightarrow{P} 1.$$

Now (5.4) will hold if both

$$(5.5) \quad \Sigma\, Z_i^2 \xrightarrow{P} 1$$

and

$$(5.6) \quad (c_n\ \lambda_n)^{-1}\ (\hat{\beta}_n-\beta)^2\ \Sigma(N_n(I_i))^2 \xrightarrow{P} 0$$

The following lemma shows that the Z_i may be assumed independent in proving (5.5).

<u>Lemma 5.1</u> Assume the conditions of Lemma 3.1 for $\psi_1(x)=x_+$, and $\psi_2(x)=1_{(x>0)}$, and let $\psi(x) = \psi_1(x) - \beta\psi_2(x)$. Let $\lambda_n^{(1)}, \lambda_n^{(2)}, \lambda_n$ be defined as in (2.3) relative to ψ_1, ψ_2, ψ, respectively and let (5.2) hold. Then (with Section 2 notation) $\Sigma\, Z_i^2 - \Sigma\, U_i^2 \xrightarrow{P} 0$. It then follows that (5.5) holds if it holds with the Z_i assumed independent.

<u>Proof</u>: With $V_i = Z_i - U_i$ we have

$$(5.7) \quad \Sigma\, Z_i^2 - \Sigma\, U_i^2 = 2\Sigma\ V_iZ_i + \Sigma\ V_i^2.$$

Defining $V_{ni}^{(1)}$, $V_{ni}^{(2)}$ with respect to ψ_1, ψ_2 as V_{ni} is defined relative to ψ, we have

$$V_{ni} = (\lambda_n^{(1)}/\Lambda_n)^{\frac{1}{2}}V_{ni}^{(1)} - \beta(\lambda_n^{(2)}/\Lambda_n)^{\frac{1}{2}}V_{ni}^{(2)}$$

so that

$$\begin{aligned}\mathcal{E}\Sigma\, V_{ni}^2 &= k_n\mathcal{E}V_{n1}^2 \le K\Lambda_n^{-1}k_n(\lambda_n^{(1)}\mathcal{E}V_{n1}^{(1)2} + \lambda_n^{(2)}\mathcal{E}V_{n1}^{(2)2}),\\ &\le Kc_n\ell_n/(n\lambda_n) + o(1),\end{aligned}$$

by (3.7), applied to ψ_1 and ψ_2. This tends to zero by (5.2) and the basic assumption $k_n\ell_n/n \to 0$. Hence $\Sigma\, V_i^2 \xrightarrow{P} 0$. Further $\mathcal{E}\Sigma\, Z_i^2$ is bounded, by a similar argument, using (3.5) and (5.2) so that ΣZ_i^2 is tight and hence $|\Sigma\, V_iZ_i| \le (\Sigma\, V_i^2)^{\frac{1}{2}}(\Sigma\, Z_i^2)^{\frac{1}{2}} \xrightarrow{P} 0$. The first statement thus follows from (5.7) and the second by the argument used in Lemma 2.1 since e.g.

$$|\mathcal{E}\exp(it\Sigma\, U_j^2) - \Pi\mathcal{E}\exp(itU_j^2)| \le 16k_n\, \alpha_{n,\ell_n} \to 0. \qquad \blacksquare$$

The main result of this section now follows readily. In this $m_k = m_{n,k}$ will be used to denote the kth central moment $\mathcal{E}(Z_1-\mathcal{E}Z_1)^k$ of

$$Z_1 = (c_n\lambda_n)^{-\frac{1}{2}}\sum_{j=1}^{r_n}\psi(X_j-u_n)_+ .$$

<u>Theorem 5.2</u> Let F satisfy (4.3). Let the basic assumptions and (2.5) hold for $\psi_1(x) = x_+$, $\psi_2(x) = 1_{(x>0)}$, and write $\psi(x) = \psi_1(x)-\beta\psi_2(x)$. Let $\lambda_n^{(1)}, \lambda_n^{(2)}, \lambda_n$ be defined as in (2.3) relative to ψ_1, ψ_2, ψ respectively and suppose that $\lambda_n^{(k)} \le K\Lambda_n$, k=1,2, n=1,2,3,... . Assume that $c_n/\Lambda_n \longrightarrow \infty$, $k_nm_{n,4} \to 0$, and (5.2) holds both as stated and with $\lambda_n^{(2)}$ replacing λ_n. Then $\hat{\lambda}_n/\Lambda_n \xrightarrow{P} 1$ and hence (4.23) holds with $\hat{\lambda}_n$ replacing λ_n.

<u>*Proof*</u>: As noted above it is sufficient to show that (5.5) and (5.6) both hold.

Write $\mathcal{E}Z_{n1}=m$. By (3.4) applied to ψ_1 and ψ_2 it is readily seen that

$$(5.8) \qquad k_n|m| \le K(c_n/\lambda_n)^{½}$$

so that $k_n m^2 \longrightarrow 0$ by (5.2), and hence $\mathcal{E}(\Sigma Z_i^2) = k_n(\mathrm{var}Z_{n1}+m^2) = 1+o(1)$. Thus (5.5) clearly follows if it is shown that $\mathrm{var}(\Sigma Z_i^2) \to 0$. Now assuming independence of the Z_i by Lemma 5.1, it is readily checked that

$$\mathrm{var}\ \Sigma Z_i^2 = k_n\ \mathrm{var}\ Z_1^2 \le k_n(m_4+4m_3m + 4m_2m^2).$$

The first term $k_n m_4$ tends to zero by assumption. The second is dominated by $4k_n\ m_4^{3/4}\ m = o(1)\ k_n^{1/4}\ m$ which tends to zero by (5.8) and (5.2). Since $m_2=1/k_n$ the final term is $4m^2$ which also tends to zero by (5.8) and (5.2). Hence (5.5) follows.

Finally to show (5.6) note (defining $Z_{n1}^{(2)}$ as Z_{n1} but with respect to ψ_2) that

$$\begin{aligned}\mathcal{E}(c_n\lambda_n)^{-1}\ \Sigma(N_n(I_i))^2 &= k_n(\lambda_n^{(2)}/\lambda_n)\mathcal{E}Z_{n,1}^{(2)2} \\ &\le K(1 + c_n/(\lambda_n^{(2)}k_n))\end{aligned}$$

by (3.5) and the assumed boundedness of $\lambda_n^{(2)}/\lambda_n$. Hence it follows from (5.2) with $\lambda_n^{(2)}$ for λ_n that the means of the random variables $(c_n\lambda_n)^{-1}\ \Sigma(N_n(I_i))^2$ are uniformly bounded, and hence these r.v.'s form a tight sequence. Since $\hat{\beta}_n \xrightarrow{P} \beta$ (cf remark after (4.14)), (5.6) now follows.

■

6. Tail and quantile estimators

An important reason for interest in the estimators from the previous section is estimation of small tail probabilities and large quantiles. For example quantiles are important for design of engineering structures and tail probabilities give the reliability of existing structures.

Thus, the problem is to use observations $X_1, \ldots, X_n$ to estimate probabilities or quantiles which are well outside "the range of the sample" so that non-parametric methods do not apply. Our starting point will be the tail condition (3.1), viz.

$$\text{(6.1)} \qquad \frac{1 - F(x+t)}{1 - F(t)} \to e^{-x/\beta}, \; t \to \infty, \; x \in \mathbb{R}.$$

To obtain the estimators we will just assume equality in 6.1, and replace β by $\hat{\beta}_n$ and 1-F(z) by $N_n(z)/n$, with $N_n(z)$ the number of exceedances of z, as before. In this Section we will only consider the choice $z = X^{(n)}_{c_n}$, for sequences $\{c_n\}$, with $c_n \to \infty$, $c_n/n \to 0$, although the results from Section 4 indicate this is not crucial.

Sometimes interest is not in tails and quantiles of the observations themselves, but in the corresponding quantities for maxima over some period, of length N, say. For example, in the water level data studied in Section 8 below, measurements are taken twice daily, at high tide, but the code stipulates that the probability of flooding the dike during a year should not exceed 1/10,000. Thus, the quantity needed is the p-th quantile of the yearly maxima, for p=1/10,000. To obtain estimators, we will assume an extremal index $\theta > 0$ exists, as discussed in the introduction, so that

$$\text{(6.2)} \qquad \begin{aligned} P(M_N > z + x) &\approx 1 - F(z+x)^{N\theta} \\ &\approx 1-\exp\{-N\theta(1-F(z+x))\} \\ &\approx 1-\exp\{-N\theta(1-F(z))e^{-x/\beta}\}, \end{aligned}$$

in the range considered. Estimators are again obtained by assuming equality and replacing θ by $\hat{\theta}_n$ and β by $\hat{\beta}_n$, 1-F(z) by $N_n(z)/n$.

Thus far, we have discussed four cases, i.e. estimation of tails and quantiles for individual random variables and for maxima. There is also a fifth case which we will comment on briefly, when N is much larger than n, and one wants to estimate the distribution of M_N. We will treat the four cases separately, the ideas each time being the same but the details somewhat different.

The discussion will be in terms of exponential-type tails satisfying (6.1). However, an extension to regularly varying tails is only a matter of straightforward translation. This is briefly discussed in the next section.

1. *Tail estimation* As outlined above, the tail probability

$$p = p_n = 1-F(y),$$

for $y=y_n$ increasing with n, is estimated by

$$\hat{p}_n = \frac{c_n}{n} \exp\{-(y-X_{c_n}^{(n)})/\hat{\beta}_n\} \tag{6.3}$$

$$= \frac{c_n}{n} \exp\{-W_n/\hat{\beta}_n\},$$

for

$$W_n = y - X_{c_n}^{(n)}.$$

A simple "propagation of errors" calculation for var $\{e^{-A/\hat{\beta}_n}\}$ in terms of the mean and variance of $\hat{\beta}_n$ suggests that the asymptotic variance of $\hat{p}_n$ can be estimated by

$$\hat{\lambda}(\hat{p}_n) = n^{-2} W_n^2 c_n \hat{\beta}_n^{-4} \hat{\lambda}_n e^{-2W_n/\hat{\beta}_n} \tag{6.4}$$

$$= W_n^2 \hat{\beta}_n^{-4} \hat{\lambda}_n c_n^{-1} \hat{p}_n^2$$

for $\hat{\lambda}_n$ given by (5.1). We will prove asymptotic normality when $n\to\infty$, $y = y_n \to \infty$, $np_n \to 0$. The last condition in particular means that $w_n \to \infty$, for $w_n = y - u_n$.

where u_n $(=U(n/c_n))$ satisfies (4.4), as before. Further, put

$$g(t) = 1/\beta - F'(t)/(1-F(t)).$$

<u>Theorem 6.1</u> Suppose the conditions of Theorem 4.3 are satisfied and that $np_n \to 0$. If furthermore

$$\sqrt{c_n/\hat{\lambda}_n} \sup_{t\geq 0} g(t+u_n/\beta) \xrightarrow{P} 0, \; n \to \infty, \tag{6.5}$$

and

$$W_n^2\hat{\lambda}_n/c_n \xrightarrow{P} 0, \tag{6.6}$$

then

$$\hat{\lambda}(\hat{p}_n)^{-1/2} (\hat{p}_n - p_n) \xrightarrow{d} N(0,1). \tag{6.7}$$

<u>Proof</u> It follows from $\sqrt{c_n/\lambda_n} \to \infty$, (4.19) and Theorem 3.5 that $W_n - w_n = u_n - X_{c_n}^{(n)} \xrightarrow{P} 0$, and since $np_n \to 0$ implies $w_n \to \infty$, also $W_n/w_n \xrightarrow{P} 1$. Further, it then follows from Theorem 4.3 and (6.6) that

$$\frac{W_n}{\hat{\beta}_n} - \frac{w_n}{\beta} \xrightarrow{P} 0, \quad n \to \infty.$$

Hence it is sufficient to prove asymptotic normality with W_n replaced by w_n and $\hat{\beta}_n$ replaced by β in (6.4).

Now, write

$$F_n = \sqrt{c_n/\lambda_n}\, E_n(X_{c_n}^{(n)}) = \sqrt{c_n/\lambda_n}\, (\hat{\beta}_n(X_{c_n}^{(n)}) - \beta_n).$$

$$G_n = \sqrt{c_n/\lambda_n}\, (X_{c_n}^{(n)} - u_n).$$

so that $F_n \xrightarrow{d} N(0,1)$ and G_n is tight, by Theorem 4.3, and Lemma 4.2 and (4.19). Then,

$$\hat{p}_n = \frac{c_n}{n} \exp \{- (\frac{w_n}{\beta_n} - \frac{\sqrt{\lambda_n}\, G_n}{\beta_n \sqrt{c_n}}) \ / \ (1 + \frac{\sqrt{\lambda_n}\, F_n}{\beta_n \sqrt{c_n}})\}$$

$$\sim \frac{c_n}{n} e^{-w_n/\beta} \{ 1 + \frac{\sqrt{\lambda_n}\, F_n w_n}{\beta^2 \sqrt{c_n}} + \frac{\sqrt{\lambda_n}\, G_n}{\beta \sqrt{c_n}} \},$$

since F_n and G_n are tight, and since

$$\beta_n \xrightarrow{P} \beta, \quad \frac{\lambda_n}{c_n} \to 0, \quad w_n^2 \frac{\lambda_n}{c_n} \to 0,$$

by assumption, where $\sim$ means that the ratio of the two sides tends to one in probability. Further, since $c_n/n = \overline{F}(u_n)$, $p_n = \overline{F}(u_n + w_n)$,

$$\frac{c_n}{n} e^{-w_n/\beta} - p_n = \frac{c_n}{n} e^{-w_n/\beta} (1 - e^{w_n/\beta} \frac{\overline{F}(u_n+w_n)}{\overline{F}(u_n)}.$$

Hence

$$(6.8) \quad \hat{\lambda}(\hat{p}_n)^{-1/2}(\hat{p}_n - p_n) \sim \sqrt{\frac{\lambda_n}{\hat{\lambda}_n}} \{F_n + \frac{\beta G_n}{w_n}\} + \frac{\beta^2 \sqrt{c_n}}{w_n \sqrt{\hat{\lambda}_n}} \left\{ 1 - \frac{e^{u_n/\beta} \ \overline{F}(u_n+w_n)}{e^{u_n/\beta} \ \overline{F}(u_n)} \right\}$$

with $\overline{F} = 1 - F$, as before. Since G_n is tight, $w_n \to \infty$, and $\lambda_n/\hat{\lambda}_n \xrightarrow{P} 1$ by Theorem 5.2, it follows that the first term is asymptotically standard normal.

It thus only remains to prove that the last term tends to zero. By Taylor's formula (cf. the proof of $B_n \longrightarrow 0$ in Lemma 4.1), this term is asymptotically equivalent (in the sense that the ratio of the two expressions tends to one in probability) to

$$- \sqrt{\frac{c_n}{\hat{\lambda}_n}} \ \frac{\beta^2}{w_n} \log \frac{e^{w_n/\beta} \ \overline{F}(u_n+w_n)}{\overline{F}(u_n)} = - \beta \sqrt{\frac{c_n}{\hat{\lambda}_n}} \int_0^1 g(u_n+sw_n)ds$$

$$\xrightarrow{P} 0.$$

by (6.5). ■

<u>Remark 6.2</u> (i) Since $\hat{\lambda}_n/\lambda_n \xrightarrow{P} 1$ and $W_n/w_n \xrightarrow{P} 1$, (6.5) and (6.6) might as well have been stated with $\hat{\lambda}_n$, W_n replaced by λ_n, w_n. However, (6.5), (6.6) have the advantage that they involve only observed quantities, except for the function g.

(ii) From the form of $\hat{\lambda}(\hat{p}_n)$ and (6.6) it follows that the relative error $\hat{p}_n/p_n - 1$ tends to zero.

2. *Quantile estimation* Let $x_p = U(1/p)$ be the $(1-p)$-th quantile (assumed to be unique) of the marginal d.f. F of the X_i's, so that $F(x_p) = 1-p$. Reasoning as before, x_p may be estimated by

$$\hat{x}_p = \hat{\beta}_n \log \frac{c_n}{np} + X^{(n)}_{c_n}, \tag{6.9}$$

for $p = p_n \longrightarrow 0$, with $np_n \longrightarrow 0$. This time, the asymptotic variance is estimated by

$$\hat{\lambda}(\hat{x}_p) = (\log \frac{c_n}{np})^2 \hat{\lambda}_n/c_n. \tag{6.10}$$

<u>Theorem 6.3</u> Suppose the conditions of Theorem 4.3 are satisfied, $np_n \longrightarrow 0$, and that F satisfies A4 of the appendix. If furthermore

$$\frac{\sqrt{c_n/\hat{\lambda}_n}}{\log(c_n/np_n)} \alpha(u_n) \xrightarrow{P} 0, \tag{6.11}$$

with $\alpha(u_n)$ as in A4

$$\hat{\lambda}(\hat{x}_p)^{-1/2} (\hat{x}_{p_n} - x_{p_n}) \xrightarrow{d} N(0,1). \tag{6.12}$$

<u>Proof</u> With the same notation as in the proof of Theorem 6.1,

$$(6.13)\qquad \frac{\sqrt{c_n\hat{\lambda}_n}}{\log(c_n/np_n)}(\hat{x}_{p_n} - x_{p_n}) = \{ F_n + \frac{G_n}{\log(c_n/np_n)}\}$$

$$- \frac{\sqrt{c_n\hat{\lambda}_n}}{\log(c_n/np_n)}\{x_{p_n} - u_n - \beta_n \log \frac{c_n}{np_n}\} .$$

Here the first term is asymptotically standard normal, since $F_n \xrightarrow{d} N(0,1)$, G_n is tight, and $c_n/np_n \longrightarrow \infty$.

To prove that the second term tends to zero, first note that $-\log(1-F(u_n)$ $= \log(n/c_n)$ by (4.4). Let V be the right-continuous inverse of $-\log(1-F(u_n))$ and $a(t) = \alpha(V(t))$, as in the proof of Lemma A1. Then, for $\tilde{a}(t) = a(\log(t))$, we have that

$$\begin{aligned}\tilde{a}(\frac{n}{c_n}) &= \alpha(V(\log(\frac{n}{c_n}))) \\ &\sim \alpha(u_n),\end{aligned}$$

and hence

$$(6.14)\qquad \frac{\sqrt{c_n\hat{\lambda}_n}}{\log(c_n/np_n)}\tilde{a}(n/c_n) \longrightarrow 0, \quad n \longrightarrow \infty$$

Further, $x_{p_n} = U(1/p_n)$, $u_n = U(n/c_n)$, so that the second term in (6.13) may be written as

$$(6.15)\qquad \int_1^{c_n/np_n} \{s \frac{n}{c_n} U'(s \frac{n}{c_n}) - \beta\} \frac{ds}{s} \le 2d \int_1^{c_n/np_n} \tilde{a}(sn/c_n) \frac{ds}{s}$$

by the appendix (Lemma A2) and the following remark. Now A9 which follows from A4 implies that for sufficiently large n and $\epsilon > 0$, with ρ as in A3

$$\tilde{a}(sn/c_n) \le \tilde{a}(n/c_n)(1+\epsilon)s^{-\beta\rho+\epsilon},$$

uniformly for $s \geq 1$ ([5] Proposition 1.7.5). Hence (6.15) is bounded by

$$2d\tilde{a}(\frac{n}{c_n}) \int_1^{c_n/np_n} (1+\epsilon)s^{-\beta\rho+\epsilon} \frac{ds}{s}.$$

Together with (6.14) this shows that the second term in (6.13) tends to zero in probability if A3 and A4 hold. ■

Remark 6.4 (i) Note that for the exponential distribution A4 holds with $c=0$, any $\rho>0$, and any α satisfying A8

(ii) Similarly to Remark 6.2, $\hat{\lambda}_n$ may be replaced by λ_n in (6.11).

3. *Estimation of the tail of M_N* For this we need an estimator $\hat{\theta}_n$, say, for the extremal index $\theta>0$ (assumed to exist). An example of such an estimator (studied in [8]) is

$$\hat{\theta}_n = \hat{\theta}_n(z_n) = \frac{1}{N_n(Z_n)} \sum_{i=1}^{r_n} \eta_i(z_n),$$

where $\eta_i(z_n) = 1$ if there is at least one exceedance of Z_n by the X_j's for j in the i-th block, J_i, and zero otherwise. As before $z_n = X_{c_n}^{(n)}$ is the natural choice. Further, let

$$\theta_n = \frac{k_n}{c_n} \mathcal{E}\eta_1(u_n),$$

with c_n, u_n as in (4.4), and define an auxilliary quantity $\tilde{\theta}_n = \tilde{\theta}_n(y,N)$ by assuming equality in (6.2), i.e. assume that

$$P(M_N > y) = 1-\exp\{-N\tilde{\theta}_n(1-F(y))\}$$

$$= 1-\exp\{-N\tilde{\theta}_n p_n\},$$

so that $\tilde{\theta}_n \longrightarrow \theta$. Here we will not confine ourselves to some special form of

$\hat{\theta}_n$, θ_n, but will only assume we have some estimators $\hat{\theta}_n$ and constants θ_n which satisfy

$$\theta_n \to \theta > 0,$$

(6.16) $$\frac{\sqrt{c_n/\hat{\Lambda}_n}}{W_n} \, (\hat{\theta}_n - \theta_n) \xrightarrow{P} 0,$$

and (6.20) below.

The obvious estimate of $p_n(N) = P(M_N > y)$, for $y = y_n \to \infty$, is

(6.17) $\hat{p}_n(N) = 1 - \exp\{-N \, \hat{\theta}_n \, \hat{p}_n\}$,

for $\hat{p}_n$ given by (6.3). Its variance may be estimated by

(6.18) $\hat{\lambda}(\hat{p}_n(N)) = N^2 \, \hat{\theta}_n^2 (1 - \hat{p}_n(N))^2 \, \hat{\lambda}(\hat{p}_n)$,

with $\hat{\lambda}(\hat{p}_n)$ given by (6.4).

In the case when $p_n(N)$ is small one may, by Taylor's formula, use the alternative estimator

$$\hat{\hat{p}}_n(N) = N \, \hat{\theta}_n \, \hat{p}_n,$$

and estimate its variance by

$$N^2 \, \hat{\theta}_n^2 \, \hat{\lambda}(\hat{p}_n).$$

<u>Theorem 6.5</u> Suppose $N\hat{p}_n \xrightarrow{P} 0$, (6.16) holds, and the conditions of Theorem 6.3 are satisfied. Then

(6.19) $\hat{\lambda}(\hat{p}_n(N))^{-1/2} \, (\hat{p}_n(N) - P(M_N > y)) = T_n + \hat{\lambda}(\hat{p}_n)^{-1/2} \, \hat{\theta}_n^{-1} p_n(\theta_n - \tilde{\theta}_n)$,

where $T_n \xrightarrow{d} N(0,1)$. If in addition

(6.20) $$\frac{\sqrt{c_n/\hat{\Lambda}_n}}{W_n} \, (\theta_n - \tilde{\theta}_n) \xrightarrow{P} 0,$$

then

$$\hat{\lambda}(\hat{p}_n(N))^{-1/2} \, (\hat{p}_n(N) - P(M_N > y)) \xrightarrow{d} N(0,1).$$

<u>Proof:</u> By Remark 6.3 (ii), $\hat{p}_n/p_n \xrightarrow{P} 1$, and hence $Np_n \to 0$ and $\tilde{\theta}_n \to \theta$, $\tilde{\theta}_n \to \theta$ imply that

$$N(\hat{\theta}_n\hat{p}_n - \tilde{\theta}_n p_n) \xrightarrow{P} 0, \quad n \to \infty. \tag{6.21}$$

Thus, by the definition of $\tilde{\theta}_n$,

$$\begin{aligned}\hat{p}_n(N) - P(M_N > y) &= (1-\hat{p}_n(N))(\exp\{N(\hat{\theta}_n\hat{p}_n-\tilde{\theta}_n p_n)\} - 1)\\ &\sim (1-\hat{p}_n(N))N\{\hat{\theta}_n(\hat{p}_n-p_n) + (\hat{\theta}_n-\theta_n)p_n + (\theta_n-\tilde{\theta}_n)p_n\},\end{aligned}$$

and hence

$$\begin{aligned}\hat{\lambda}(\hat{p}_n(N))^{-1/2}(\hat{p}_n(N) - P(M_N>y)) \sim\ & \hat{\lambda}(\hat{p}_n)^{-1/2}(\hat{p}_n-p_n) + \hat{\lambda}(\hat{p}_n)^{-1/2}\,\hat{\theta}_n^{-1}\, p_n(\hat{\theta}_n-\theta_n)\\ &+ \hat{\lambda}(\hat{p}_n)^{-1/2}\,\hat{\theta}_n^{-1}\, p_n(\theta_n-\tilde{\theta}_n).\end{aligned}$$

It follows from (6.16), $\hat{\beta}_n \to \beta$, and

$$\hat{\lambda}(\hat{p}_n)^{-1/2}\, p_n = \frac{p_n}{\hat{p}_n}\sqrt{\frac{c_n}{\hat{\lambda}_n}\,\frac{\hat{\beta}_n^2}{W_n}} \sim \sqrt{\frac{c_n}{\hat{\lambda}_n}\,\frac{\hat{\beta}_n^2}{W_n}} \tag{6.22}$$

that the second term on the righthand side tends to zero in probability. Setting

$$T_n = \hat{\lambda}(\hat{p}_n)^{-1/2}(\hat{p}_n-p_n)$$

now proves (6.19). Since $\hat{\theta}_n \xrightarrow{P} \theta > 0$ by (6.16), the last result of the theorem follows at once from (6.19) and (6.20). ■

<u>Remark 6.6</u> (i) To establish (6.20) is a separate problem in probability theory, and clearly depends on which particular process one is considering. However, from a practical point of view, (6.20) requires that N is large compared to typical cluster sizes. Of course, (6.17) should only be used when this is the case.

(ii) The assumption $Np_n \to 0$ (or, equivalently, that $N \hat{p}_n \xrightarrow{P} 0$) is only used for (6.21). However, (6.21) obviously is also satisfied in more general circumstances, and $\hat{p}_n(N)$ seems useful also for N such that $P(M_N > y_n)$ does not tend to zero. This corresponds to the fifth case mentioned in the beginning of this section. ∎

4. *Estimation of quantiles of M_N* Assume that the distribution of M_N has a unique (1-p)-th quantile $x_p(N)$ given by

$$p = P(M_N > x_p(N)).$$

The straightforward estimator of $x_p(N)$ then is

$$\hat{x}_p(N) = \hat{\beta}_n \log \frac{c_n N \hat{\theta}_n}{n \log(1/(1-p))} + X_{c_n}^{(n)}$$

and an estimator of its asymptotic variance is

$$\hat{\lambda}(\hat{x}_p(N)) = \left[\log\left\{ \frac{c_n N \hat{\theta}_n}{n \log(1/(1-p))} \right\}\right]^2 \hat{\lambda}_n / c_n .$$

For $p=p_n$ small one may alternatively use

$$\hat{\hat{x}}_p(N) = \hat{\beta} \log \frac{c_n N \hat{\theta}_n}{n\, p} + X_{c_n}^{(n)},$$

$$\hat{\lambda}(\hat{\hat{x}}_p(N)) = \left[\log\left\{ \frac{c_n N \hat{\theta}_n}{n\, p} \right\}\right] \hat{\lambda}_n / c_n .$$

<u>Theorem 6.7</u> Suppose the conditions of Theorems 6.3 and 6.5 are satisfied for p_n replaced by $(\log(1/(1-p)))/N$ (implying in particular that $np_n/N \longrightarrow 0$). Then

(6.23) $$\hat{\lambda}(\hat{x}_p(N))^{-\frac{1}{2}} (\hat{x}_p(N) - x_p(N)) = T_n + \hat{\lambda}(\hat{x}_p(N))^{-\frac{1}{2}} \hat{\beta}_n \log \frac{\hat{\theta}_n}{\tilde{\theta}_n} ,$$

where $T_n \xrightarrow{d} N(0,1)$. If furthermore

(6.24) $\hat{\lambda}(\hat{x}_p(N))^{-\frac{1}{2}} (\hat{\theta}_n - \tilde{\theta}_n) \xrightarrow{P} 0,$

then

$$\hat{\lambda}(\hat{x}_p(N))^{-\frac{1}{2}} (\hat{x}_p(N) - x_p(N)) \xrightarrow{d} N(0,1).$$

Proof: Define the function

$$\pi(\theta) = \frac{\log(1/(1-p))}{N\theta}, \quad 0 < \theta \leq 1.$$

Then

$$x_p(N) = x_{\pi(\tilde{\theta}_n)},$$

with x_p the (1-p)-th quantile of F, as before. Further, comparing with (6.9) we have that

$$\hat{x}_p(N) = \hat{x}_{\pi(\hat{\theta}_n)}.$$

Hence

$$\hat{x}_p(N) - x_p(N) = (\hat{x}_{\pi(\hat{\theta}_n)} - \hat{x}_{\pi(\tilde{\theta}_n)}) + (\hat{x}_{\pi(\tilde{\theta}_n)} - x_{\pi(\tilde{\theta}_n)})$$

$$= \hat{\beta} \log \frac{\hat{\theta}_n}{\tilde{\theta}_n} + (\hat{x}_{\pi(\tilde{\theta}_n)} - x_{\pi(\tilde{\theta}_n)}).$$

Now, since $\tilde{\theta}_n/\hat{\theta}_n \xrightarrow{P} 1$, it follows from Theorem 6.3, by straightforward arguments, that

$$T_n = \hat{\lambda}(\hat{x}_p(N))^{-\frac{1}{2}} (\hat{x}_{\pi(\tilde{\theta}_n)} - x_{\pi(\tilde{\theta}_n)})$$

$$\xrightarrow{d} N(0,1), \qquad n \to \infty,$$

which proves (6.23). Since $\hat{\beta}_n \xrightarrow{P} \beta$, (6.24) implies that the last term in (6.23) tends to zero in probability, which concludes the proof of the theorem. ■

7. Regularly varying tails

We now very briefly indicate how the results should be translated when the tail 1-F decreases in a "polynomial" (regularly varying), rather than exponential, manner. More specifically condition (3.1) is replaced by

$$(7.1) \qquad \frac{1-F(xt)}{1-F(t)} \longrightarrow x^{-1/\beta}, \qquad t \longrightarrow \infty,\ x > 0.$$

Clearly, if a positive random variable X has distribution function F satisfying (7.1) then the distribution function of logX satisfies (6.1) with the same β. The following condition replaces (4.3) for the present case,

$$(7.2) \qquad \frac{tF'(t)}{1-F(t)} \longrightarrow \frac{1}{\beta} \qquad \text{as } t \longrightarrow \infty.$$

For the convenience of the reader, we reformulate some of the results of Sections 4 and 6 for distribution functions satisfying (7.1) or (7.2).

Define now (cf. (4.1))

$$(7.3) \qquad \hat{\beta}_n(z_n) = \{\sum_{i=1}^{n} (\log X_i - z_n)_+\} / \{\sum_{i=1}^{n} 1_{\{\log X_i \geq z_n\}}\}$$

$$= \frac{1}{N_n(z_n)} \sum_{i=1}^{n} (\log X_i - z_n)_+ \ ,$$

where N_n in the present section has been redefined as

$$N_n(x) = \sum_{i=1}^{n} 1_{\{\log X_i > x\}}.$$

Let λ_n be as in (2.3) and $\hat{\lambda}_n$ be as in (5.1), but with X_i replaced by $\log X_i$. Theorem 4.3 may be immediately restated in the present context. For example if the tail condition (7.2) holds and the other conditions of Theorem 4.3 are satisfied with X_i replaced by $\log X_i$, then

$$(7.4) \qquad \sqrt{\frac{N_n(z_n)}{\lambda_n}}(\hat{\beta}_n(z_n) - \beta_n) \xrightarrow{d} N(0,1).$$

for $z_n = u_n$ and $z_n = \log X^{(n)}_{c_n}$ where $\beta_n = \frac{n}{c_n} \mathcal{E}(\log X_1 - u_n)_+$. Similarly Theorem 5.2 may be simply adapted to give a result under which (7.4) holds with $\hat{\lambda}_n$ replacing λ_n.

Next consider the analogue of Theorem 6.1. In addition to the specifications above let

$$\begin{aligned} g(t) &= \frac{1}{\beta} - \frac{e^t F'(e^t)}{1-F(e^t)} \\ p_n &= 1-F(e^y) \end{aligned} \tag{7.5}$$

and let $\hat{p}_n$ and W_n be as in (6.3) but with X replaced by log X. Then the formulation of the Theorem 6.1 goes through without further changes.

To adapt Theorem 6.3, replace A4 by

$$\lim_{t\to\infty} \frac{\frac{tx\, F'(tx)}{1-F(tx)} - \frac{1}{\beta}}{\gamma(t)} = cx^{-\rho} \tag{7.6}$$

with $\gamma > 0$ satisfying

$$\lim_{t\to\infty} \frac{\gamma(tx)}{\gamma(t)} = x^{-\rho} \tag{7.7}$$

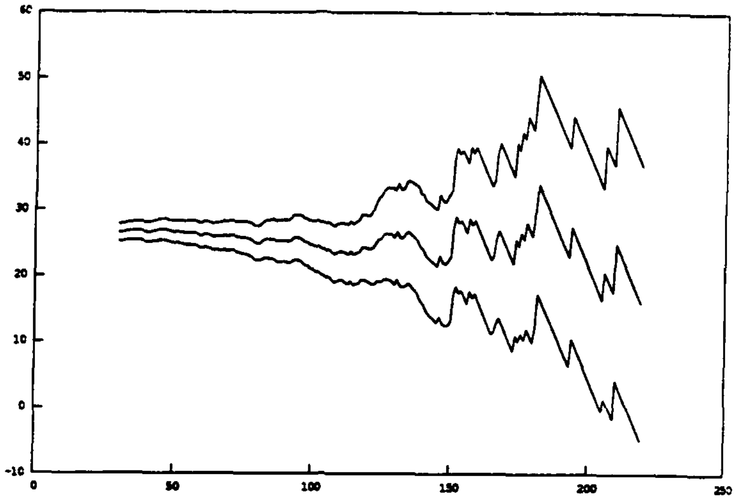
Let $x_p = \log U(1/p)$ and let $\hat{x}_p$ be as in (6.9) but with X replaced by log X. Then the formulation of Theorem 6.3 goes through, with $\alpha(t)$ replaced by $\gamma(e^t)$.

8. Application to water level data.

Reliable high tide water levels are available from about 1885 onwards at five stations along the Dutch coast. We restrict ourselves to the station Hoek van Holland (part of the city of Rotterdam) and observe that high water levels are mainly due to wind storms. All data obtained outside winter periods, October 1 - March 15, are removed: significant wind storms mainly occur during the winter.

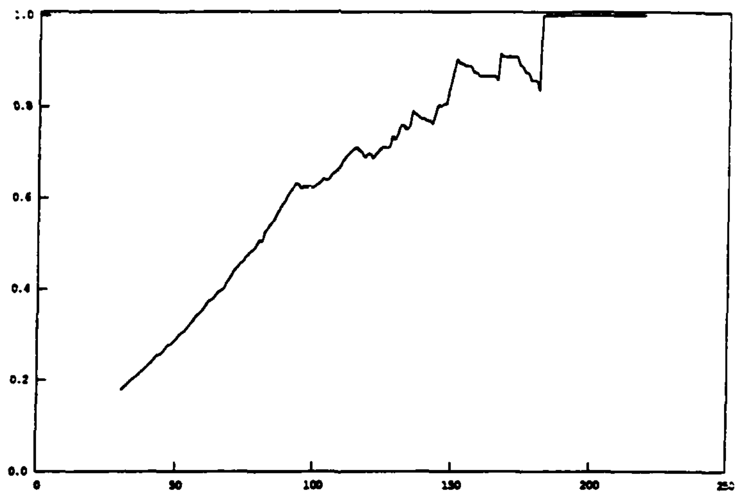
One is interested in the tail of the marginal distribution, in view of the design of sea dikes. Since high levels are mainly due to wind storms, there is short range dependence in the data - the influence of a severe wind storm typically lasts several days - but not much long range dependence. The theory of clustering of high values and extremal indices seems suitable for description of the available data. Also at first glance the exponential distribution gives a reasonable fit and the data seems stationary. Since extrapolation outside the range of observations is quite critical, it seems wise to consider a larger class of distributions than just the exponential one, so we adopt assumption A1. In order to single out the influence of wind storm activity we did not use the original observations but so-called set up values, that is the difference between the observed value and the value predicted on the basis of the movements of sun, moon and earth ("astronomical levels").

In this way a data set of size 17,544 covering the years 1887-1985 is obtained. The estimates $\hat{\beta}_n$ and $\hat{\theta}_n$ were calculated (cf. (4.1) and (6.2)) for various levels u_n and the 95% two-sided confidence interval for β obtained. Figure 8.1 shows β_n and its estimated confidence interval against the chosen levels u_n. The blocksize $r_n = 30$ has been used for the intervals (c.f. formula (5.1)). However, the intervals were rather insensitive to changes in r_n. As expected the value of β_n fluctuates substantially when u_n is high, since then few observations are used. From a theoretical point of view a bias could develop when u_n is low (since then β_n may differ significantly from β). This phenomenon does not seem to occur in the range considered here. Figure 8.2 shows $\hat{\theta}_n$ plotted against the chosen level u_n. The approximate monotonicity of this function points towards a serious bias of the estimation method for low levels. However, θ_n is clearly less than one.



u(cm)

Fig. 8.1 Estimates $\hat{\beta}_n=\hat{\beta}_n(u)$ and approximate 95% confidence intervals for β, based on n=17,544 tide level measurements at Hoek van Holland (r_n=30).



u(cm)

Fig. 8.2 Estimates $\hat{\theta}_n=\hat{\theta}_n(u)$ of the extremal index θ for the same tide level measurements as in Fig. 8.1.

9. Simulations

To assess the behaviour of $\hat{\beta}$ for small samples the following processes were simulated,

$$\left.\begin{array}{ll}\text{EMA0:} & X_t = e_t \\ \text{EMA1:} & X_t = e_t + e_{t+1}\end{array}\right\} \begin{array}{l}\{e_t\}\text{ i.i.d.} \\ \text{exponential r.v.'s}\end{array}$$

$$\left.\begin{array}{ll}\text{PMA1:} & X_t = e_t + e_{t+1} \\ \text{PAR1:} & X_t = (.958X_{t-1} + e_t)/1.95\end{array}\right\} \begin{array}{l}\{e_t\}\text{ i.i.d. Pareto} \\ \text{r.v.s with } P(e_t>x)=1/x \\ \text{for } x > 1.\end{array}$$

The first two of these processes have asymptotically exponential tails, and the last two have asymptotically Pareto tails. All four have $\beta=1$, and the θ-values are 1, 1, 0.50, 0.51.

The simulation for the PAR1-process was started with X = 0, discarding the first 500 values. To give an extra check on the results, several of the simulations were independently programmed and run twice, using different standard random number generators. For each replication the quantity

$$V = \sqrt{\frac{N(z_n)}{\hat{\lambda}_n}}\,(\hat{\beta}_n - \beta_n)$$

was computed, for the first two processes from (4.1), (4.2) and (5.1), and for the PAM1 and PAR1-processes from the formulae in Subsection 6.5, using the logarithms of the observations. In the simulations (except for fig. 9.1b)) the sample size was n = 4000 and c_n was chosen as np for p = .1 and p = .05, i.e. c_n was 400 or 200. Here, we used the value 4000 to yield c_n's for which the deviations from the normal limit still are quite clear.

All the simulations were performed both for "fixed c_n" and "fixed u_n". However, as was expected, the differences between the two cases were quite small, and hence only the "fixed c_n" results are exhibited below.

According to Theorems 5.2 and 4.3 (ii), V should be approximately normally distributed. The results of the simulations are given in Tables 9.1, 9.2, and

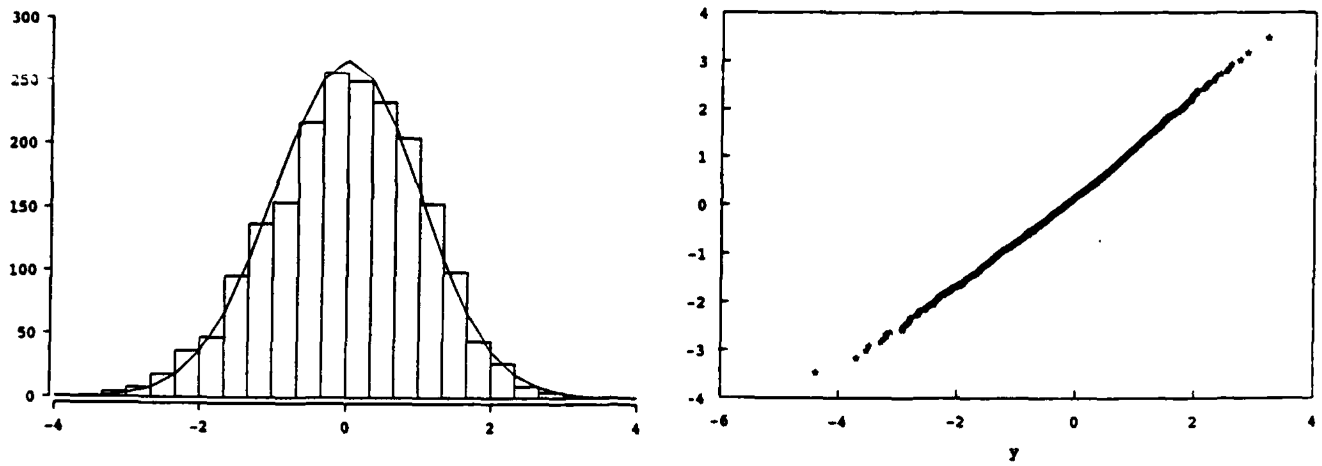
Figure 9.1 below.

	(c_n=np) p	$\nu_n(-1.64)$	$\bar{\nu}(1.64)$	$\nu(-1.96)$	$\nu(1.96)$
EMAO	.1	.07	.04	.04	.02
	.05	.08	.03	.05	.01
EMA1	.1	.07	.04	.05	.02
	.05	.08	.04	.05	.01
PMA1	.1	.10	.02	.06	.01
	.05	.08	.03	.05	.02
PAR1	.1	.14	.04	.09	.01
	.05	.16	.02	.12	.01
Normal probability		.05	.05	.025	.025

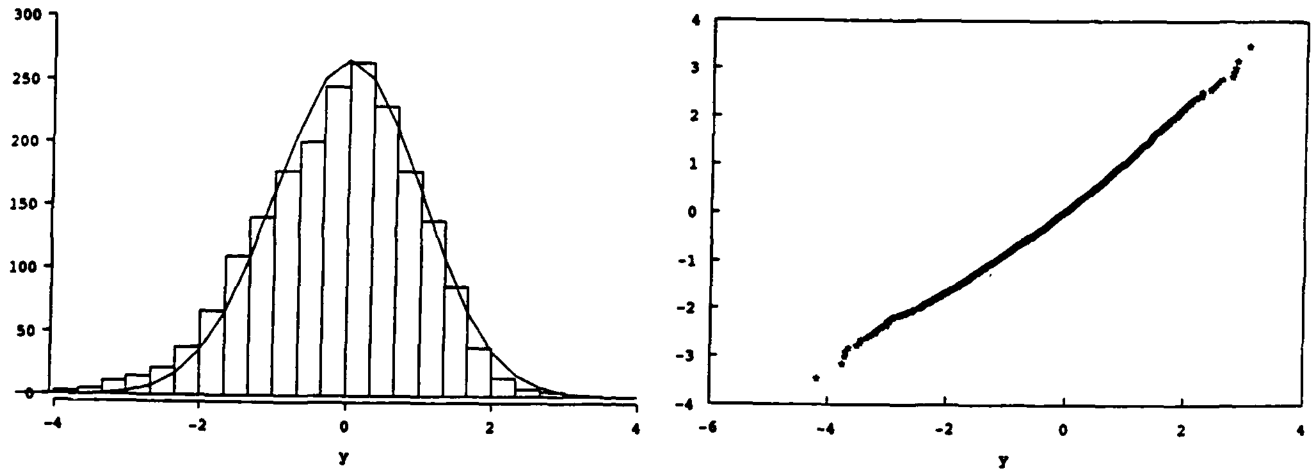
Table 9.1. Values of $\nu(x)$ = #{simulations with $V \leq x$}/2000 and $\bar{\nu}(x) = 1-\nu(x)$ based on 2000 simulations of each of the eight cases, for sample size n=4000, fixed c_n, and block size r_n = 20.

block size r	$\nu(-1.64)$	$\bar{\nu}(1.64)$	$\nu(-1.96)$	$\nu(1.96)$
2	.13	.04	.08	.02
4	.11	.03	.07	.01
10	.10	.03	.06	.01
20	.10	.02	.06	.01
40	.10	.03	.06	.01
Normal probability	.05	.05	.025	.025

Table 9.2 Values of $\nu(x)$ = #{simulations with $V \leq x$}/2000 and $\bar{\nu}(x)=1-\nu(x)$ based on 2000 simulations of each of the seven cases, for the PMA1-processes, with sample size n=4000 and fixed c_n = np for p = .1, i.e. c_n = 400.



(a) n = 8,000



(b) n = 4,000

Fig. 9.1 Histograms and normal probability plot of V-values from 2000 simulations of each of the sample sizes n = 4000 (Fig. 9.1.a) and n = 8000 (Fig. 9.1.b) for the PMA1-process. In the simulations c_n was fixed, and equal to np for p=.1, the block size was r_n = 20, and the smooth curve in the histograms is the standard normal density.

From the results above it can be seen that even for c_n = 400 and n = 4,000 there are clear deviations from the limit. A main reason for this is variability, and to some extent bias, of the λ_n-estimator. In addition, the stronger dependence in the PAR1-process also seems to slow down convergence.

For $c_n = 800$, say, (and $n = 8000$) the normal fit is much better as seen from Fig. 9.1.

It is clear that the block sizes $r_n=2$ and $r_n=4$ are too small for the PMA1-process (c.f. Table 9.2). However, there does not seem to be much difference between $r_n = 10$, 20 or 40, even if one also looks at the values of $\hat{\lambda}$. This is as expected, since $\theta = .51$, and hence clusters on the average contain about two exceedances.

Appendix Tail conditions

We use three types of tail conditions which are increasingly more restrictive.

a. (Domain of attraction condition) Suppose

A1 $$\lim_{t\to\infty} \frac{1-F(t+x)}{1-F(t)} = e^{-x/\beta} \qquad (x \in \mathbb{R})$$

for some positive constant β.

b. (Smoothness condition) Suppose F has a derivative F' and

A2 $$\lim_{t\to\infty} \frac{F'(t)}{1-F(t)} = 1/\beta \ .$$

c. (Second order condition) Suppose there exists a positive function α satisfying

A3 $$\lim_{t\to\infty} \frac{\alpha(t+x)}{\alpha(t)} = e^{-\rho x} \qquad (x \in \mathbb{R})$$

for some positive constant ρ such that

A4 $$\lim_{t\to\infty} \frac{\dfrac{F'(t+x)}{1-F(t+x)} - 1/\beta}{\alpha(t)} = c\ e^{-\rho x} \qquad (x \in \mathbb{R})$$

for some real constant c.

Remark Note that A4 implies A3 if $c \neq 0$.

Lemma A1 A4 implies

$$\lim_{t\to\infty} \frac{\frac{1-F(t+x)}{1-F(t)} - e^{-x/\beta}}{\alpha(t)} = -c\, e^{-x/\beta} \cdot \frac{1-e^{-\rho x}}{\rho} \quad (x \in \mathbb{R})$$

locally uniformly.

Proof By A1 (which follows from A4) with $H := -\log(1-F)$

$$\frac{e^{x/\beta}\, \frac{1-F(t+x)}{1-F(t)} - 1}{\alpha(t)} \sim - \frac{H(t+x) - H(t) - x/\beta}{\alpha(t)}$$

$$= - \int_0^x \left[\frac{F'(t+s)}{1-F(t+s)} - \frac{1}{\beta} \right] \frac{ds}{\alpha(t)} \to -c \int_0^x e^{-\rho s}\, ds$$

by A4 and [5, Theorem 1.8 (ii)] (or [4], for c=0). The local uniformity follows from the local uniformity in A4.

Lemma A_2

1. $c \Rightarrow b \Rightarrow a$.

2. Let V be the (right continuous) inverse function of $-\log(1-F)$. Equivalent forms of a,b and c are

a'. Suppose

A5 $$\lim_{t\to\infty} V(t+x) - V(t) = \beta x \quad (x \in \mathbb{R})$$

for some positive constant β.

b' Suppose V has a derivative V' and

A6 $$\lim_{t\to\infty} V'(t) = \beta .$$

c' Suppose there exists a positive function a satisfying

A7 $$\lim_{t\to\infty} \frac{a(t+x)}{a(t)} = e^{-\beta\rho x} \qquad (x \in \mathbb{R})$$

for some positive constant ρ, such that

A8 $$\lim_{t\to\infty} \frac{V'(t+x) - \beta}{a(t)} = de^{-\beta\rho x} \qquad (x \in \mathbb{R})$$

for some real constant $d = -\beta^2 c$.

Proof 1. The implications are immediate since any function $\alpha(t)$ satisfying A3 converges to zero $(t \to \infty)$.

2. For the equivalence of a and a', see [[5] Proposition 1.7(9)]. The equivalence of b and b' is immediate.

c<=>c': First note that A3 and A4 together are equivalent to A3 and A4 with x=0 and similarly A7 and A8 are equivalent to A7 and A8 with x=0. Let $a(x)=\alpha(V(x))$ and let $V^{\leftarrow}$ be the right-continuous inverse of V (so that typically $V^{\leftarrow} = -\log(1-F)$). Since

$$\frac{1}{V'(V^{\leftarrow}(t))} = \frac{F'(t)}{1-F(t)} .$$

for x=0 the left hand side of A4 equals

$$\lim_{t\to\infty} \frac{-1}{\beta V'(V^{\leftarrow}(t))} \frac{V'(V^{\leftarrow}(t))-\beta}{a(V^{\leftarrow}(t))} = \frac{-1}{\beta^2} \lim_{t\to\infty} \frac{V'(V^{\leftarrow}(t))-\beta}{a(V^{\leftarrow}(t))} = \frac{-1}{\beta^2} \lim_{s\to\infty} \frac{V'(s)-\beta}{a(s)}.$$

Since the equivalence of A3 and A7 is immediate from a', it now follows that c holds if and only if c' holds. □

Remark. (i) Let $U = (1/(1-F))^{\leftarrow}$, so that $U(t) = V(\log t)$. Then A6 at once translates into $tU'(t) \to \beta$, and A8 into

A9 $$\lim_{t\to\infty} \frac{t\, U'(tx)-\beta}{a(\log t)} = dx^{-\beta\rho}$$

(ii) Let $X_1, X_2, \ldots$ be i.i.d. with distribution function F. Condition a is equivalent to

$$P\{\max(X_1,\ldots,X_n) - U(n) \leq x\} \to \exp\{-e^{-x/\beta}\}$$

$(n \to \infty)$ for all x. Condition b implies that

$$\frac{\sqrt{c_n}}{\beta}\left\{X_{c_n}^{(n)} - U\left(\frac{n}{c_n}\right)\right\}$$

has asymptotically a standard normal distribution, where $c_n \to \infty$, $c_n/n \to 0$ and $\{X_i^{(n)}\}_{i=1}^n$ are the descending order statistics of $X_1,\ldots,X_n$. Condition c (cf. Lemma A_1) is sufficient for the asymptotic normality of Hill's estimator, (cf. [3] Theorem 3.1 and Remark 4).

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